Tribology

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Lecture No. # 25

Estimating Elastic Deformation

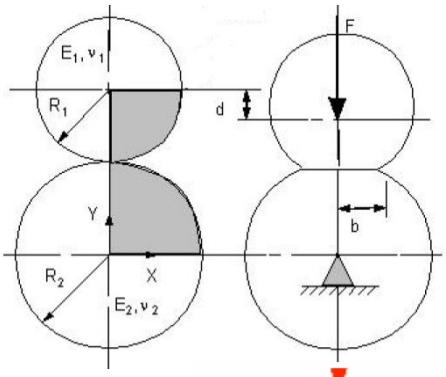
Welcome to twenty fifth lecture of video course on Tribology. So, we are going to complete today quarter century of the lectures on Tribology. Title of the topic is estimating elastic deformation. It was started long time back. When materials come in contact there will be elastic deformation, plastic deformation. We are restricting our discussion to elastic deformation, we do not want plastic deformation, we do not want permanent deformation on the surface it should recover back once the load is removed or after unloading.

In previous lecture we started with highdro dynamic lubrication and explore how to model hydrodynamic lubrication between the materials and in present lecture we are going to combine elastic deformation with hydro dynamic action. That is what name of the topic may be also can be termed or can be called as elasto hydro dynamic lubrication. Here we are going to mix elastic deformation of the materials, hydro dynamic reaction due to velocity and lubrication because of presence of lubricants.

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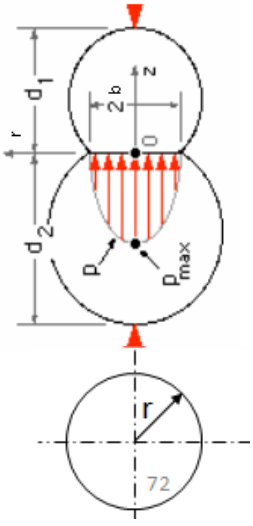
Spherical contact

$$p = p_{\max}\sqrt{\left[1-\left(\frac{r}{b}\right)^2\right]}$$



Total applied load on contact patch is $F = \int_0^b \int_0^{2\pi} p\, r\, d\theta\, dr$

Total applied load on contact patch is $F = 2\pi \int_0^b p_{\max}\sqrt{\left[1-\left(\frac{r}{b}\right)^2\right]}\, r\, dr$



I am just going to repeat for the continuity couple of the few slides which we discussed in previous lecture. Say that this is figure which is showing the two spheres they are in contact and of course the circles are as shown as a front view of those two spears in contact. And appears initially that there is a point of contact, but, if you apply the node f, point contact cannot be sustained. There would be infinite stresses and we know there will not be infinite stresses there will be some sort of deformation, material deform to sustain the load and that deformation in bigger the scale is shown over here. Of course this deformation will be very less.

That is why we can say that the half width if I say 2 b total of deformation which is shown out here or we say this is b and this is symmetrical on both the sides and this b must be very low compared to d 1. Or we say b need to be lesser than d 1. It may even be lesser than one percent. It does not reach even the one percent. Just re-sketch it. So that we can demonstrate a distribution we have kept in a bigger scale or we say that these figures are not on this scale. It is just the demonstration is a schematic diagram.

And when this kind of deformation occurs, you can see there will be contact area and that contact area in this case particularly is sphere versus sphere will be circular. Or there is some sort of miss alignment, it may be electrical, but, in this mostly this is circular and we are assuming that this radius starts from 0 to max value of b where the maximum deformation occurs. And external reports indicate that pressure will be parabolic. Maximum value will be at the centre or whether the load is coming to the maximum pressure will occur there and it will gradually decrease or maybe say parabolic decrease, not gradually, parabolic is decreased to zero value and the extreme bond is b or when r is equal to b in this pressure will be 0.

Now, what is our interest? To find out how much elastic deformation really has happened or what is this contact area. Now, there are two variables; we say that it is readily compressed as well as flattened also. So, we have two. The d is a deformation and b is a width of the contact or we say that this is a circular area, r will be variable as well as this deflection will be variable. For convenience, we start with parabolic distribution we say the pressure at any radius will be equal to p maximum. That will occur at the center with r is equal to 0 and a square root of 1 minus r by b square r is going to vary from 0 to b. When r is zero, pressure is going to be maximum. When r is equal to b; that means, b by b it will be 1 minus 1 will be equal to 0; that means, pressure is equal to 0. And this is what we are showing through this

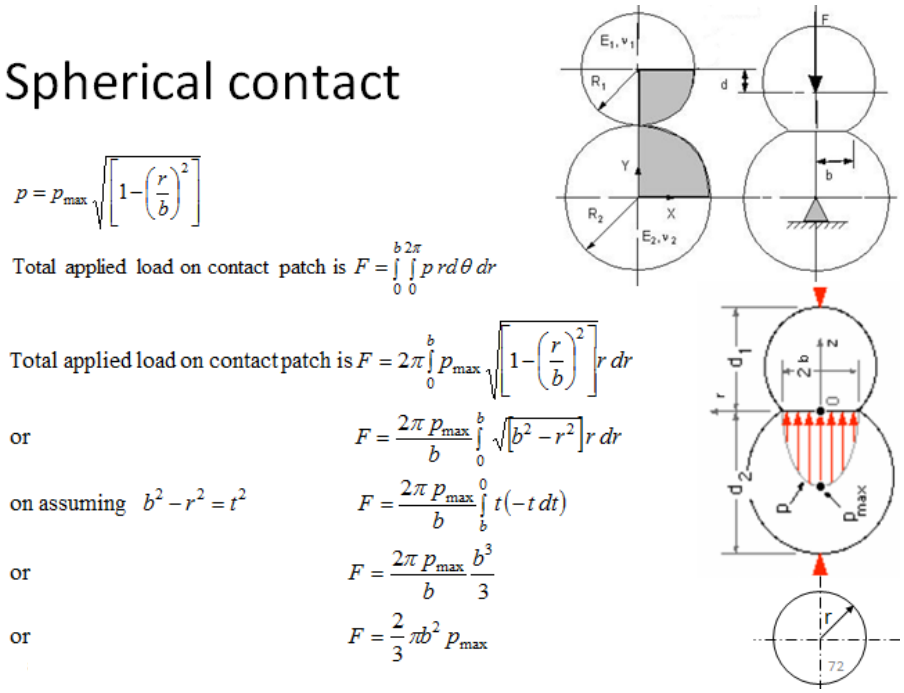
diagram. Pressure will be 0 at the boundaries and maximum at the center. That means, this kind of relation is representing the pressure distribution which is generally observed in experimental results.

Now, a common thing given to us is the force. How much force is applied on a material of course within the elastic limits. We have young's modules and knew that poison ratio as well as geometry radius. So, these two lasting properties; this is the geometric property, and the material property we need to account this. And force is generally given to us naturally all the parameters like a related to b or deformation can be figured out once we find out the expression for the F which are that the total load on can be represented as a integration of p or that the p into d a and integrate all the compete area. What is the complete area? You say this is circular. So, at any point if the starting is 0 theta is equal to 0 it will compete at as a theta is equal to 2 pi.

That's why it says that first integration limit for theta is 0 to 2 pi and second is a radius 1 and this radius is going to vary from 0 to maximum value or the b. This assumption, the maximum width is b or radius maximum value of the b radius maximum value of the r is equal to b. So, that is why this integration is 0 to b. Now, p we are suppose assuming that it is equal to the p max square route of 1 minus r by b square. We can substitute over here and we can integrate.

However, we can see that there is no theta in this p. That means, p r d r can be kept separately and this first integration can come out with the d theta and integration will be theta and we can substitute value of the theta that is the 2 pi minus 0. It will turn out to be two pi and that is shown over here.

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The force F is equal to 2 pi. We have already integrated out theta substitute in the values of the boundary 0 and 2 pi and the 2 pi minus 0 equal to 2 pi. That is a 2 pi over here p max. Now, we can rearrange the p max also does not depend on r now. We can take it out and we can rearrange in addition we have to rearrange this bracket also. We are saying that now it is b square minus r square or we say that we are taking 1 by b as common from this which is also constant for given force, given geometry, given elastic limits, b will remain constant. So, this bracket remains which need to be integrated that is b square minus r square, limits 0 to b. We know this is simple integration. We can substitute b square minus r square is equal to some variable. Differentiate it and substitute this value.

For the time being assuming the b square minus r square is t square. If you differentiate it will turn out to be minus 2 r d which is here r d r we have to substitute and this side will turn out to be 2 t d t. That means, we can substitute easily this is t and r d r is given as a minus t d t. Now, this integration if it is simple. So, t square integrating it will turn out to be t cube divided by 3 and we can substitute the values of this d t integration limit is been changed here an integration limit was a 0 to b while this is a negative side. So, that is why the integration limit is turning out to be different b and 0.

However we say that when b is equal r, t will turn out to be 0 that is been shown over here and when b is equal to 0, t can turn out to be plus minus p. We are neglecting the term minus b because there is no negative geometry in this. It has to be only b cannot negative b. That is

why we are writing something like that 0 to b integration. Now, as I mentioned that t square integration is simple now at the t cube by 3. That is shown over here t q by 3 another substituting the values b instead of t. It turns out to be b. So, it is showing results at b cube by 3. After sending it to sign will change because of this limit turn out to be 0 minus b cube by 3 and negative will cancel all or will turn out to be positive. That is where overall expression turns out to be this.

However, there is a common one term that is a b here and b cube here we can simplify and we can express this relation something like this. So, after you applied force is equal to 2 by 3 66 percent into pi that is constant b square p max. Now, many times we know what is the p_{max}. What will be maximum compressive force which can be applied on this a never lasting body or at which pressure it will be started to forming. We can substitute these values. However, if we do not know this value then we need to use some other equation also.

Now, I will say that we know force, but, they are two variable b and p max. If I use elastic properties and material; I know that b p max what will be value of p max again to simply find out what will be the b. If I do not know with surety what will be p max then I have to use some other relation. I have to try with using some other relation. However in this case we have discussed about this spherical contact.

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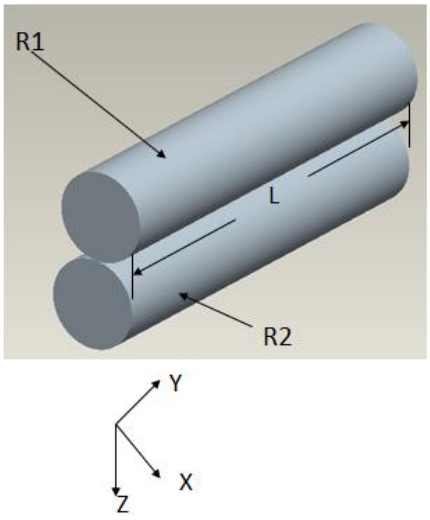
Cylindrical Contact

$$p = p_{max}\sqrt{\left[1-\left(\frac{x}{b}\right)^2-\left(\frac{y}{a}\right)^2\right]}$$

$$p = p_{max}\sqrt{\left[1-\left(\frac{x}{b}\right)^2\right]}$$

Total applied load on contact patch is

$$F = 2L\int_0^b p_{max}\sqrt{\left[1-\left(\frac{x}{b}\right)^2\right]}\,dx$$



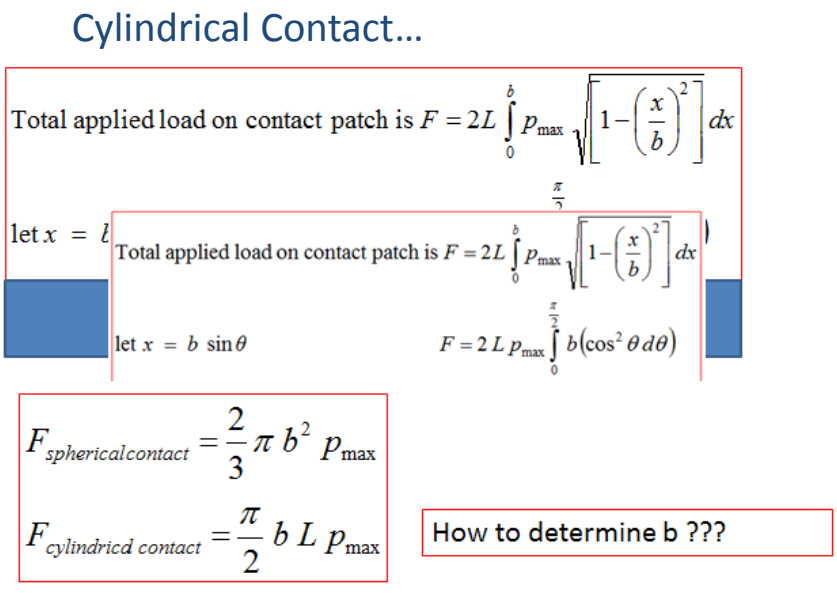
The second general contact is a cylindrical contact. That is shown in this slide. You say that in the cylinder what you we are seeing the cylinder because in this case the length is much

larger more than four times of the diameter and radius is given in R 1 and R 2. Both the cylinders have the same length. It is a typical example of the roller bearing. In roller bearing many times we keep alpha d is one, but, in this case we have assuming this, much larger. That may be L more than four times compared to the diameter and pressure distribution because there are two components, x component and y component. We can give pressure distribution something like this. Again the p will be the maximum at the center and x as direction shows here. As x varies it will vary and when it reaches to maximum value of the contact patch that is equal to b.

Pressure will not be 1 minus 1 or 0. Whatever the value remaining here, that will be shown out, can be represented. However, this was our assumption. This length is much larger, the pressure distribution or pressure variation along the y axis may not be very dominant. So, we may neglect this for the time being. You can say that y by a square term is negligible compared to x minus x divided by b squared. If we neglect this and pressure term will turn out to be the most simpler in the way in which we have solved in a previous slide. Same thing. We will as in previous slide, we use a word letter r instead of x. In this case we consider polar coordinate system.

Now, the p is known to us. We can substitute in a force equation and integrate that force equation to find out what will be the relation between the F, b and p max. So, when we do this? Here d y the way we have done earlier. The d theta this same thing in the d y will be simply integrated it turn out to be L and here it is 0 to b p max 1 by 1 minus x by b square b x. p max is also constant it can be taken out easily. And then we can substitute something x is equal to b sin theta, b cos theta in some other term and get integration done.

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For the time being in this case, we are substituting x = b sin θ So, when you substitute x as b here, b will cancel out. It will now turn out to be 1 equals to sin θ. When you find out it will turn out to be Cosθ only. When integrated d x is equal to it will turn out to be b cos theta. So, overall this term turning out to be the cos square theta. We know because integration of cos theta is simpler than integration of cos square theta. It can be termed as a cos 2 theta plus 1 divided by 2 and we can integrate because we have integration limit given over here.

So, when we integrate cos 2 theta and integration limits are 0 in pi by 2; the overall integration will turn out to be 0 because sin 2 theta is substituted by 2, substitute 0 both the values will be 0. I have a plus 1 which will come out to be the cos 2 theta plus 1 and plus 1 can be integrated easily. That will turn out to be theta. And when you substitute this boundary condition or this integration limit, it will turn out to be pi by 2. We can substitute those values and this overall expression turn out to be in terms of pi b L P_{max}.

Now, again the way we have treated a spherical contact, we can similarly, code the same thing about cylindrical contact. We say that if we know the P_{max} I can find out what will be the b value. L is already known geometry. This is a constant P_{max} if I know material, I can find the contact patch width. When we apply load F for the some materials. Now, we can compare this two we say the spherical contact with this cylindrical contact. You can see this two that there is a 2 by 3 and here it is one half. So, the 66 percent this will be 50 percent pi, pi is a common.

While b, b 1 b is common, but, second b there is b while it is a L. Length is generally very large compared to the b or more than the 10 times and p max if the material is the same the p max will remain same. So, what is the conclusion coming out from this comparison? We can say that as b is much lesser compared to L and this difference is not very high. This is 66 percent. This is 50 percent and the difference is not very high, this is 16 percent difference. But, b will be ten times. So, we say that more than ten times the 6.6 while in this case the b will b one time, but, l will be ten times. So, we can say this is a 0.66 and this will turn out to be 5. So, 5 verses 0.66. In other words that cylindrical contact can bear much larger load compared to the spherical contact.

So, then when there is a larger load limit we need to carry out with their larger load; we need to carry larger load it should sit from a spherical contact with the cylindrical contact. We know very well the friction force will increase with a spherical contact, cylindrical contact. In a spherical contact friction force or friction resistance will be lesser compared to cylindrical

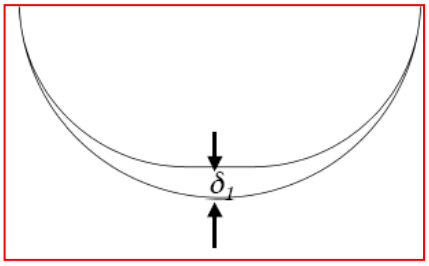
contact. But, load carrying capacity is higher. So, which is an advantage if you we have space restriction we know that load dimensions are permitted that along the length direction, but, lesser dimension is permitted over longer dimension or diametrical direction and we can choose a cylindrical contact. Of course these things we will discuss when we talk about the roller bearing or selection of the rolling element bearings.

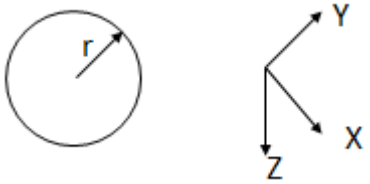
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Elastic Deformation suggested by Timoshenko & Goodier:

$$\delta_1 = \frac{\left(1-\nu^2\right)}{2\pi E}\frac{F}{r}$$

$$\delta_1(r,\theta) = \frac{1-\nu_1^2}{2\pi E_1}\int_0^b\int_0^{2\pi}\frac{p_{\max}\sqrt{1-(r/b)^2}}{r}\,r\,d\theta\,dr$$





Ref: S. Timoshenko and J.N.Goodier, Theory of elasticity, 2nd Edition, McGraw Hill.

The present terms may be I do not know the p max and then I do not know b. How to find out this b, when we do not know the p_{max}? So, we can start with some material equation. We assume this is spherical contact and under deformation this delta one is the deformation. When we are applying the force there is a deformation over here. This can be represented by an applied force, geometry or material parameter. What is the radius? If I keep equal to b I should be able to find out what will be the relation.

However, if I do not know f as it is and if I know the pressure distribution the way we have done earlier and previous example. In terms of pressure we can find out in that term. Now, why we are emphasizing on the pressure because when we discuss about the hydro dynamic lubrication; we had not discussed in terms of force we have discussed in terms of pressure distribution. So, we will know what will pressure distribution after solving the hydro dynamic equation.

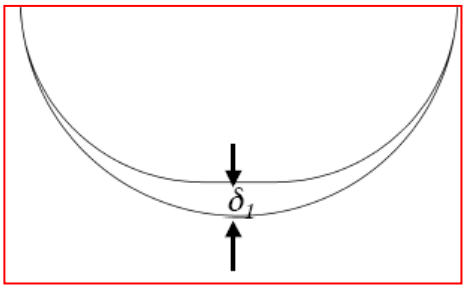
However, if we do not require hydro dynamic we require simple lasting deformation then we can treat directly f. We do not have to come to the pressure. What we required elastic deformation to combined, to be joined with hydro dynamic lubrication. That is why we are talking about the pressure term. We are giving explanation term, so the pressure. So, when we do that represent f in terms of p max pressure distribution and area d r theta d r. So, this overall can be represented in terms of b now because we required relation also in reflection and extent of the zone or extent of the contact zone. And this is what the same distribution will be shown as a coordinate system and this is the formula available in number of text books. One reference is given over here.

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$$\text{or} \quad \delta_1 = \frac{1-\nu_1^2}{2\pi E_1} 2\pi \int_0^b \frac{p_{\max}\sqrt{1-(r/b)^2}}{r}\, r\, dr$$

$$\text{or} \quad \delta_1 = \frac{\left(1-\nu_1^2\right)}{E_1} p_{\max} \int_0^b \sqrt{1-(r/b)^2}\, dr$$

on assuming $r = b \sin\theta$



$$F = \frac{2}{3}\pi b^2\, p_{\max}$$

Now when we substitute this, rearrange equation what we get in integration bracket is one minus r by b square. Square root of that d r you can substitute or as a b sin theta simple to get the value of integration. When you substitute here and rearrange we are getting this term in this form of the cos 2 theta plus one. You have done earlier, but, here I have written in more explicit form. Now, when you integrate cos 2 theta it turns out to be sin 2 theta and when you put this limit 0 and pi by 2; both the terms will turn out to be zero and this is only important to integrate. Once when you integrate 1 into d theta it will turn out to be theta and you substitute this value it will turn out to be pi by 2 minus 0.

So, we can substitute this values and integrate, that is the way it is been shown here sin 2 theta divided by 2 plus theta. This is turning out to be 0 and this will turn out to be pi by 0

when you substitute this integration limits. So, now what we have? We have this deflection in terms of b, in terms of p max and we can use a previous relation, but, we have done for this spherical contact that is the 2 by 3 pi b square p max. Now, we have also p max p max and b here we can use this relation and we can find out in terms of b relation.

However there is another possibility. We can go ahead with some sort of simplification. We can use a teller series to find out exclusively what will be in terms of f, and replacing p max with other geometric parameters.

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$$\delta_1 = \frac{b\left(1-\nu_1^2\right)}{2E_1} p_{max} \frac{\pi}{2} \qquad F = \frac{2}{3}\pi b^2 p_{max}$$

$$\delta_1 = OB - OC$$

$$\text{or, } \delta_1 = R_1 - \sqrt{OA^2 - AC^2}$$

$$\text{or, } \delta_1 = R_1 - \sqrt{R_1^2 - b^2}$$

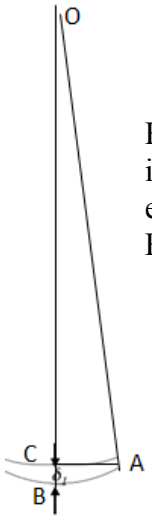
$$\text{or, } \delta_1 = R_1\left(1 - \sqrt{1-\left(\frac{b}{R_1}\right)^2}\right)$$

$$\text{or, } \delta_1 = R_1\left(1 - \left(1 - \frac{1}{2}\left(\frac{b}{R_1}\right)^2 + \text{negligible terms}\right)\right)$$

$$\text{or, } \delta_1 = R_1 \frac{b^2}{2R_1^2}$$

$$\delta_1 = \frac{3}{8b}\frac{1-\nu_1^2}{E_1} F$$

$$b^3 = 0.75 R_1 \frac{1-\nu_1^2}{E_1} F$$



How to incorporate these expressions in FDM

To do that, we can simplify it. First I am substituting p max value of the p max here I want to replace p max f and represent this deflection or radial deflection on this surface that is turning out to be radial deflection delta one as 3 by 8 divided by b. So, this is related, 1 minus mu square by E_1 f.

Now, we are considering only the deflection of the one surface. You cannot consider deflection of both the surfaces. There is a possibility the other surface is rigid enough. It does not deflect at all or if there is a deflection then we should consider delta two also. And this will be changed one minus mu square divided by E_1 all other things will remain same because the contact patch will remain same. b will remain same for the both the material it is

something like the one material will have one b, b 1 and another material with different b are coming in a contact b r making a pair. So, they will have same value of b.

Now, what do I say, that simplification can come from geometry here. We know very well that this deflection needs to be much lesser then the radius of this pair. That is why I say that OA is the radius, OB is the radius and when we deflecting it to some extent it is after deflection this sphere is getting shape of OC or there is a deflection this much inside. Now, that can be represented in terms of OB minus OC. This is a OB minus OC that is a deflection. We know OB is a radius R_1 while OC can be represented with this triangle, right angle triangle. That is a OA square minus AC square OA already we know that is R 1 and this is in terms of b that half with or to we considered with overall deformation or the extent of the contact zone of the 2 b.

So, this will turn out to be b. So, this is R_1 minus b square and we know b need to be much lesser than r one. So, we can rearrange, we are taking R_1 common from this and overall common from the brackets it turns out to be one minus square root of one minus b by R_1 square. Now we can use a Taylor series for this. It will be 1 minus half b by R_1 square plus high order term as b is much lesser then R 1. So, this term high order term will be negligible that is why we are writing here somewhere the negligible terms.

Now, the delta 1 can be represented here in terms of R and in bracket is 1 minus 1 and this minus 1 and minus 1 will turn out to be plus and this is what will be in the R_1 divided by 2 into b square divided by R_1 square. So, this is a same thing what we have mentioned R 1 b square divided by 2 R_1 square. Now, there is a one relation over here in terms of delta and b. There is another relation delta and b over here and we know the force is given to us. Elastic properties or material will be known to us, geometry of the contacting surfaces will be known, generating body will be known to us.

So, we can simply find out deflection in terms of the force or what will be the contact patch in terms of the force. That is very clear from this relation I can find out either delta or b it will not be much problem. There are 2 equations 2 variables both will be a straight forward relation to us. So, here the b cube is given as a 0.75 R_1 by E_1, which an elastic property multiplied with f. In another word what we say that if f increases b is not going to increase in the same order means, if you make force 8 times then only the b will increase by two times. The sensitivity of the deformation is much lesser in this case or we say with a slight deformation, load carrying capacity of this material will be very high. Here, b need to be

lesser than R 1 and much lesser then R 1. If any time that kind of relation is violated then we will not be able to get the results properly.

That is why we say b need to be much lesser than R_1 that you mixture whenever there is a deformation and we need to compute, if any time b is turning out to be more than 1 percent or 2 percent of R_1; we should stop it, we should go ahead with non-linear analysis with this kind of simplification will not be valid.

Now, the question comes how do we incorporate in hydro dynamic lubrications? We are talking about only the solid mechanics in this case. We talk of the deformation in terms of the force, but, how to combine with our hydro dynamic lubrication which we have discussed in previous lectures and we have more interest in that. When we cannot combine hydro dynamic lubrication with elastic deformation then only the name can be given as a elastohydro dynamic lubrication which is important as I say that is the one of the optimum tribal sphere mechanism.

If we are able to achieve elastodynamic lubrication throughout the life of the component, we cannot achieve better than that results. Hydro dynamic lubrication is not as good as an elasto dynamic lubrication. Mix lubrication is not as good as a elastodynamic lubrication. It is the only the elasto dynamic lubrication which gives the best results. And most of the high performance tribal spheres are designed based on the elasto order lubrication. Talk of the gears, talk of the rolling element bearings, talk about the follower mechanisms they have been designed based on elastohydro dynamic lubrication mechanism with the load carrying capacity is enormous with a less, much lesser area they are able to sustain much larger load.

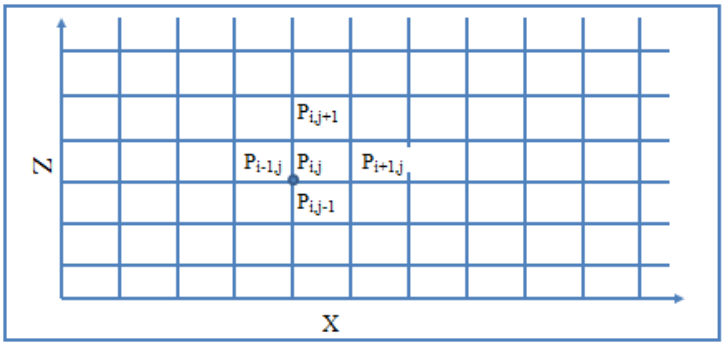
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How to incorporate Deflection in FDM

- Deformation due to a distributed normal pressure on the surface:

$$\delta = \frac{1-\nu^2}{\pi E}\iint_A \frac{p(x,z)dxdz}{\sqrt{x^2+z^2}}$$

$$\delta_{53} = \frac{1-\nu^2}{\pi E}\left[\sum \frac{p_{i,j}\,dxdz}{\sqrt{(i-5)^2dx^2+(j-3)^2dz^2}}\right]$$



Now, it is a question need to be answered how to incorporate deflection in f d m or finite difference method which was used for hydro dynamic lubrication. We derived a FDM pressure distribution or found nodal pressure. What we did in a finite difference method; we developed the surface, divided the surface in number of division we say there were number of nodes and at those number of nodes we found pressure distribution.

Now, if I know the pressure distribution at the nodes, I can integrate those pressures to find out the force. If I know the force of the difference nodes I can find out what will be the deformation because of those. However, there is a surface connectivity whole solid surface need to be integrated. It is nothing like that they are individual molecules individual pieces and they will simply deform under the load. Here we need to incorporate or combine load.

And what we are trying to convey is a deformation which we have done in earlier case and in previous slide in polar coordinates. Here we are writing a rectangular coordinates because the pressure what we have expressed in our hydro dynamic lubrications it was in terms of x and z. And. So, there is area d x into d z and this is the main thing. We say that this is the distance. What is the distance between the pressure and the deflection point with the pressure applied and what we are calculating the deflection. Now, we cannot calculate the deflection

of the same point where the pressure is been applied because that will simply penetrate, will give the infinite stress.

That is why we consider the movement equation. We take some distance from the pressure point and value of the deflection. We do not calculate the deflection of the same point now more and more accurate can be the achieved when we go ahead with more and more number of divisions. This is what I am trying to convey. See now the pressure suppose p i is somewhere here. I want to calculate what will be the fact of this pressure over this nodes this node, this node, this node and keep in mind pressure at any point is going to influence the deflection the surface with the whole surface is a connect. It is a solid surface they are separated. Think about the diaphragm when we apply a pressure at any point of the diaphragm. Whole diaphragm gets deflected. It is not only that point can get deflected, whole diaphragm gets deflected and we are going to take those in to consideration. So that, we get overall good results.

Now, just for example, I have given in this case if I want to find out the deflection at the fifth node in x direction and third node in z direction, what we are saying this is a 1 is starting is 1 1 1 2 sorry 2 1 3 1 4 1 5 1 5 1 5 3. That means, if I want to find out the word is deflection at this node due to p i j, p i j that is somewhere here. I can be anything j can be anything depends on number of divisions how will I find? Or we say the pressure p i j into area because we are taking very small areas; that means, there is no need of extra integration it will simply will be p i j into d x into d z.

That is going to give me force and this is going to give me the distance. What is the distance here? In this case what I will do, I will subtract. i minus 5 square. If there is a fifth node then, it will turn out to be 5 minus 5 that will be 0. Similarly, j minus third node that is here the pressure is a j node and minus where we want to evaluate since a minus three will be square root here and now this can clearly see when we want to find out the deflection five three due because of the pressure five three this will turn out to be infinity. When we do computer programming we mean to avoid that. If I want to find out the deflection under 5 3, I will not find due to pressure of five three whenever the pressure of 5 3 number comes, I have to simply bypass that. I do not have to add and is only it may be say 50 by 50 matrix I have to avoid one point.

See whenever that this deflection and this deflection, this node sequence and this node sequence are same, I have to avoid this summation. Remaining all other points I have to

account it as summation 50 into 50 except one. Say that 2004 100 99 pressures we will find out what will deflection because of 2004 100 and 99 pressure at this; however, if you know the boundary condition the pressure is zero at the extreme boundaries. Then, we may avoid those points also because we know the zero pressure will not cause any deflection. We can avoid the calculation of boundary, we can say start from the two go for the 49 if we have our node limit from 1 to 50. One it will give a pressure 50 again it will give zero pressure.

So, we will start from 2 to 49 in x direction as well as in z direction. So, that will reduce computational efforts. This is what we are saying that when we use this kind of a device. Now, if I already solved the pressure equation or we already solved a Reynolds equation we already know what will be the value of $p_{i,j}$ at every node. Of course we can intermediate we can put in between also when we are going through iteration loop. However this is what I showed there is $p_{i,j}$ here we calculate pressure with area near by the d x and d z and now we are trying to find out what will be the effect of this pressure force on this point and will be a number of points over here.

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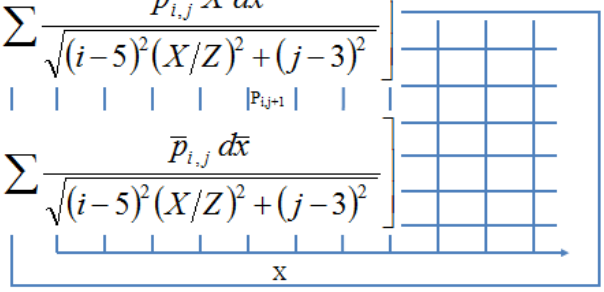
How to incorporate Deflection in FDM

$$\delta_{53} = \frac{1-\nu^2}{\pi E}\left[\sum \frac{p_{i,j}\, X\, Z\, d\bar{x}\, d\bar{z}}{\sqrt{(i-5)^2 X^2 d\bar{x}^2 + (j-3)^2 Z^2 d\bar{z}^2}}\right] \qquad \bar{p} = \frac{C^3 P}{6\eta U X^2}$$

$$\delta_{53} = \frac{1-\nu^2}{\pi E}\left[\sum \frac{p_{i,j}\, X\, d\bar{x}}{\sqrt{(i-5)^2 (X/Z)^2 + (j-3)^2}}\right]$$

$$\delta_{53} = \frac{1-\nu^2}{\pi E}\left[\frac{6\eta U X^2}{C^3}\sum \frac{\bar{p}_{i,j}\, X\, d\bar{x}}{\sqrt{(i-5)^2 (X/Z)^2 + (j-3)^2}}\right]$$

$$\bar{\delta}_{53} = \frac{1-\nu^2}{\pi E\, C}\left[\frac{6\eta U X^3}{C^3}\sum \frac{\bar{p}_{i,j}\, d\bar{x}}{\sqrt{(i-5)^2 (X/Z)^2 + (j-3)^2}}\right]$$



Now, we continue with same and when what is the difference when we were discussing about the hydro dynamic lubrication; the whole hydro dynamic lubrication equation have been represented in terms of non dimensional numbers and this is dimensional. We cannot simply combine non dimensional with the dimensional numbers. So, what we need to do; this whole expression needs to be converted to non dimensional numbers. Now to do that, instead of x d

x square we can use a non dimensional number that is a d x bar square and there will be multiplication of maximum dimension in x direction. Similarly, for the z side also instead of z, we will write a d z into z and whole square of that.

Similarly, here instead of d x we are writing x and d x bar. Instead of d z we are writing capital z as a maximum value of z in z direction into d z bar. Now, we can use a relation the way we have done in hydro dynamic lubrication. We normalize x by z we are using that bracket x by z bracket. So, I can take out this complete expression out. So, the z square, d z square take common out of this and the results are numerator z d z and here it is a square root. So, z d z bar will be cancelled out and expression will be modified something like this. You say that capital x d x bar, this a non dimensional, this is dimensional. Here it will turn out to be x divided by capital z. So, x divided by capital z is turning out to be non dimensional and we do not have any other term over here right.

Now, this is a why we have taken because we know d x bar and d z bar are equal. In hydro dynamic lubrication we took d x bar equal to d x z bar. If those are not taken equal, then we cannot take this as a common or we say that in this case we have taken both d z bar out and d x bar and d z bar both out. Assuming d x bar is equal to d z bar otherwise we will not be able to take out. As we have already done in hydro dynamic lubrication mechanism or when we are solving this equation with a non dimensional numbers or in a non dimensional form of the Reynolds equation we already took those as assumptions, we are just repeating here.

Now, this is a what? This is the whole non dimensional number I minus 5 square. I is a number x by d is a non dimensional is not going to have any dimension z minus three is a real number it is not dimensional. Now what is only dimensional here? Here the p i j is a dimensional x is a dimensional and e is dimensional. And if we try to non-dimensionalize it then, we will get deflection in terms of non dimensional number. We did normalization in hydro dynamic lubrication using this relation. We say the p bar that is the non dimensional pressure is given as p that is a dimensional pressure divided by six eta u x square by c cube. This way we non-dimensionalize the temperature we can substitute same relation here instead of p i j we can write p bar that is a non dimensional pressure into six eta u x square divided by c cube.

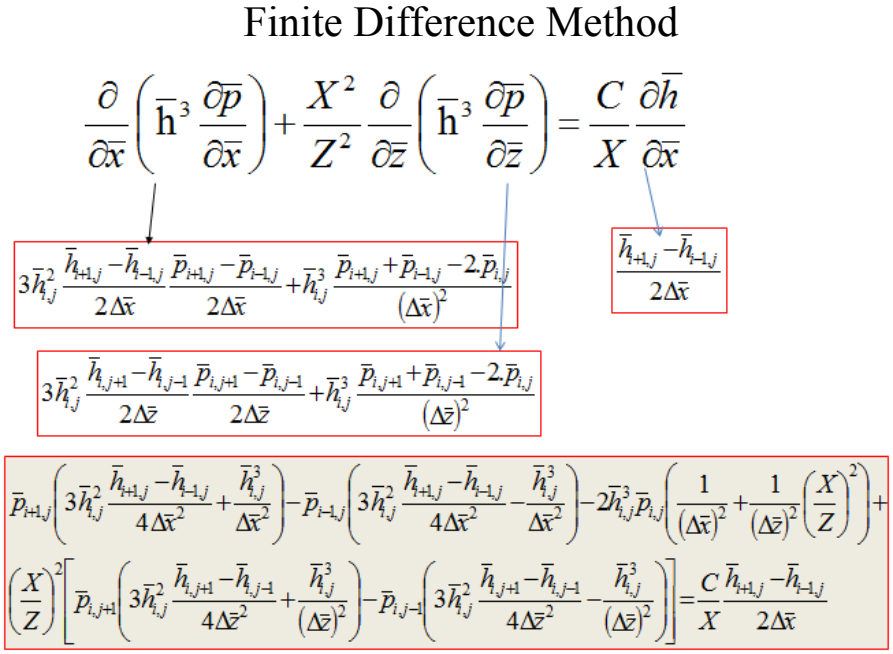
When we do that what we are getting now, this is a non dimensional number and this is a constant coming over here there is a six eta u x square divided by c cube and there is a capital x over here. That can be transferred outside. The whole summation will turn out to be non

dimensional number. That will be once we know the pressure p i j, we can simply find out what will be the deflection.

This is what I am trying to convey over here when we are moving out and further deflection this is what 6 is a number. One minus eta square is a number and when my mu square is a number, eta is a Pascal second u is a meter per unit second, the second and second will be cancelled out this will turn out to be Pascal into meter. Here it is a meter cube meter cube. So, meter cube and meter cube will be cancelled out still we have Pascal into meter and e is in Pascal. So, Pascal will be cancelled out here it will turn out to be only meter and we are dividing by c to normalize it. So, to non-dimensionalize. Meter divided by meter. So, whole of this turn out to be non dimensional number.

Now, we have converted whole dimensional deflection in non dimensional deflection. So, this equation can be simply used in film thickness expression wherever the film thickness we have calculated. We can add this expression, we can find out film thickness distribution over x direction and as well as z direction.

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Now, we need a slight modification in a finite difference method which we have done in hydro dynamic lubrication. Do you remember what we did? We converted finite difference method we use a finite difference method as center for central difference method and we evaluated h at the half node. Now, when we calculate h at the half node, deflection cannot be added in that case again we have to calculate deflection of the half node which is slightly

more cumbersome. That is why we are trying to achieve this expression. So that we know we do not require any half node calculation, we can go ahead with a full node calculation.

What is the meaning of that? Let us start with this first term there is a pressure distribution in x direction which depends on the film thickness and the pressure. What we are doing? We are doing in the parts. First we are taking differentiation of h. That will turn out to be three h square divided by dh by dx bar into dp bar by dx bar. So, first term will be that second term will be h bar cube out and this will be second differentiation of pressure.

So, we will be having two terms and we need to do central difference for the two terms. Now, you can see the first one we are only integrating first h that is the three h square three h bar square and that comes over here into d h by d x bar which is shown over here. H bar i plus 1 j minus i minus 1 j, this is what a central difference of film thickness gradient. This is a second term into not second term it is a multiplication factor d p by d x that can be evaluated here. So, something like the p bar i plus 1 j p bar i minus 1 j divided by 2 x. So, this is going to be the complete first term, second h cube h bar cube is common that is shown over here.

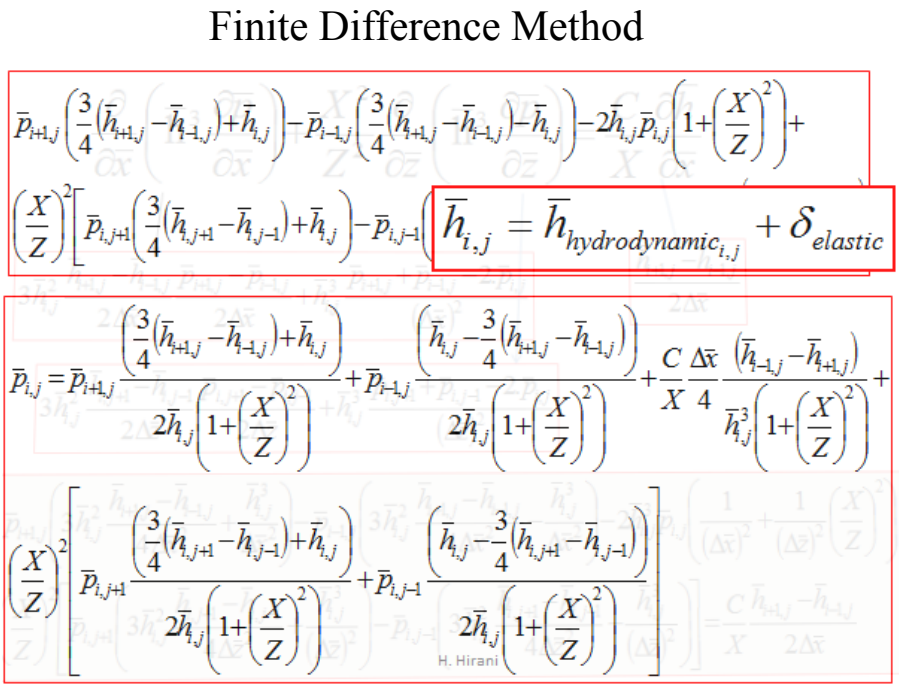
And after that this is a second derivative of the pressure that can be simply given as p bar i plus 1 plus p bar i minus 1 minus 2 p bar i j and divided by delta x square. So, this is the first expression in x direction, same thing for the z direction. This is the same thing we have to just change only I and replace I with a j and j with I; you can see here three h i j square the same. Here is h i plus 1 j instead of that here we are writing i j plus 1. Again somewhere here this is i minus 1 j, what we are writing here is i bar j minus 1. That means, we are taking variation along the z direction.

In previous case we assumed that there is no variation in film thickness along the z direction. Variation was only along the x direction, but; however, we when are calculating the pressure and pressure is deforming the surface and that will deform also in the z direction. It is nothing like the way need to avoid we need to account that also. So, this is given over here same letter is the term in this will remain same it is not much variation the way we have done a hydro dynamic lubrication same thing. Now, we substitute we get vey lengthy equation very big equation too many terms, but, we have to bear it. We do not have any escape route for that. If you want to get a solution, for this kind of expression; A what is been done? Terms have been separated, you can see p i plus one j term p i minus one j term p i j term p i j plus term p i j minus one term. Here they have been separated out and in bracket we are writing in terms of only the complete node. It is not a half node the way we have done in hydro dynamic

lubrication there was a half node while in this case we are writing complete terms complete node is a i plus 1 i minus 1 i j and similarly, j plus 1 j minus 1. There is no half term and interesting thing is that after rearranging, we are getting much lesser number of cubic terms. Cubic term is coming only in h i j we do not have any other cubic terms.

So, number of calculation will be reduced with this kind of formulation. The only problem with this kind of formulation is that it is involving more number of terms compared to the terms which we have seen in hydro dynamic lubrication mechanism and we use a finite difference method in hydro dynamic lubrication. Now, this can be rearranged because at any time we need to find out what will be $p_{i,j}$ for given value of p i plus 1 p i minus 1 p p i j plus one p i j minus 1 and the source term. This is source term.

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That's why we need to rearrange equation in such a manner we can find out p i j. That can be arranged something like this now. We had a smaller equation now is turning out to be much bigger equation right. So, because we have divided this one also 2 $h_{i,j}$ and one plus x square by z square or everywhere we have divided here, that is why the number of terms are increasing and this expression is turning out to be slightly lengthy. But, when we do computer programming it is not going to be much lengthier compared to earlier expression what we have done and we are going to get good results without much problem.

Now, first point after doing this kind of derivation, will be that let us compare with this formulation and previous formulation without accounting elastic deformation.

In this case we have not added anywhere elastic deformation for time being only thing what we have mentioned that film thickness will change.

And the deformation will be more, separation from the film thickness will increase that need to be accounted. But, yet we have not accounted here anywhere the delta 0. It should give me the same result as the hydro dynamic lubrication mechanism. If it is not giving the same result; that means, I have made some mistake in derivation I need to verify it again.

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Finite Difference Method

$$\bar{h}_{i,j} = \bar{h}_{hydrodynamic_{i,j}} + \delta_{elastic}$$

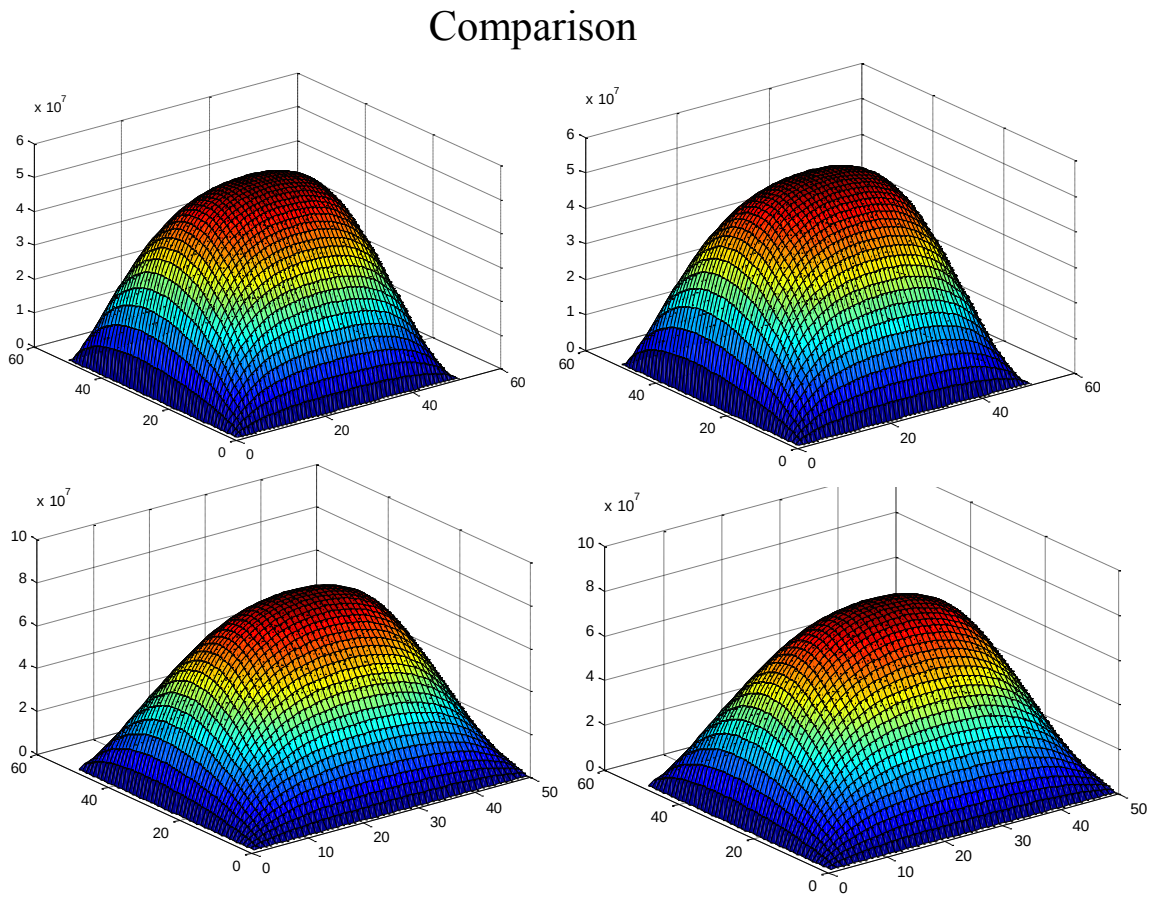
$$\bar{p}_{i,j} = \bar{p}_{i+1,j}\frac{\left(\frac{3}{4}\left(\bar{h}_{i+1,j} - \bar{h}_{i-1,j}\right) + \bar{h}_{i,j}\right)}{2\bar{h}_{i,j}\left(1+\left(\frac{X}{Z}\right)^2\right)} + \bar{p}_{i-1,j}\frac{\left(\bar{h}_{i,j} - \frac{3}{4}\left(\bar{h}_{i+1,j} - \bar{h}_{i-1,j}\right)\right)}{2\bar{h}_{i,j}\left(1+\left(\frac{X}{Z}\right)^2\right)} + \frac{C}{X}\frac{\Delta\bar{x}}{4}\frac{\left(\bar{h}_{i-1,j} - \bar{h}_{i+1,j}\right)}{\bar{h}_{i,j}^3\left(1+\left(\frac{X}{Z}\right)^2\right)} +$$

$$\left(\frac{X}{Z}\right)^2\left[\bar{p}_{i,j+1}\frac{\left(\frac{3}{4}\left(\bar{h}_{i,j+1} - \bar{h}_{i,j-1}\right) + \bar{h}_{i,j}\right)}{2\bar{h}_{i,j}\left(1+\left(\frac{X}{Z}\right)^2\right)} + \bar{p}_{i,j-1}\frac{\left(\bar{h}_{i,j} - \frac{3}{4}\left(\bar{h}_{i,j+1} - \bar{h}_{i,j-1}\right)\right)}{2\bar{h}_{i,j}\left(1+\left(\frac{X}{Z}\right)^2\right)}\right]$$

However what I am trying to convey is something like this. Here h i at any point will be given as h hydrodynamic i j. That means, whatever the film thickness was calculated for hydro dynamic lubrication plus whatever the deformation is happening and that will be also in a deformation at the i j. At every node will the deformation will be calculated and will be added in this case.

So, this will turn out to be more complex much more time consuming compared to the hydro dynamic lubrication. But, is still there is a point we can verify this full formulation keeping elastic deformation as a 0. Using this coding may be add with elastic deformation become complete code.

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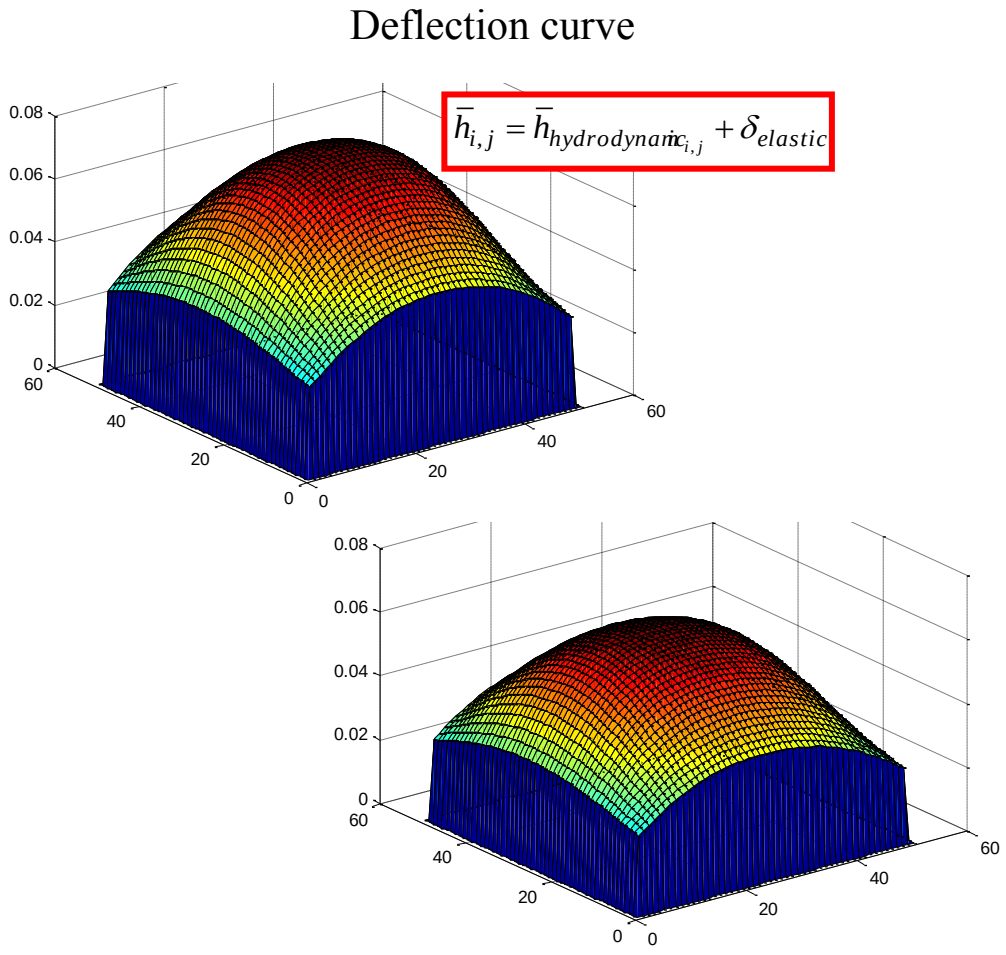
And after that put everywhere elastic deformation as 0 and we can compare this new formulation which we have done with full node without accounting half node with earlier program which we have done with a half node. Now, that this slide is going to show the comparison there is any change this is done with half node calculation. When we are counting pressure distribution and this cause effect of this viscosity on the pressure due to pressure; obviously, this viscosity thickening the first time, this viscosity thickening is not accounted while in second figure viscosity thickening is accounted.

Now, this is with a previous formulation or the half node formulation. Now, this is with a complete formulation without accounting half node. We are using only the full node. I am not able to see any difference over here. This is out this viscosity thickening is also without this viscosity thickening. This is by using the expression for the half node, this is using full node.

So, that means, we have not made any mistake in our formulation. Results are almost same not a single percent not a .1 percent change, 0.01 percent change in the results. Same thing with this also we are saying here the 4. Something, this is around forty bar pressure or we say that forty mega Pascal pressure here and the same thing over here.

So, we are not finding any difference in this curve and this curve. We here in this case have not accounted this viscosity thickening effect. While on this case this viscosity thickening effect has been accounted and we are not able to find any difference in this results. That means, what is the formulation whatever we have done in previous slides; that is all right, is not going to change results, it is not affecting the results.

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Now, the time has come to find out what will be the elastic deformation. So, this is elastic deformation. What is been done over here? We calculated. We calculate the pressure at the every node and simplify what is the deformation without going with a iteration loop, without accounting in the film thickness expression, this is been done only in once we know the pressure and find out the deflection at every node.

Now, we know the 50 by 15 node. At the initial there will be zero deflection because the pressure is 0. After that is going high and reaches to some value maximum. Now, if I compare with a non dimensional film thickness. Non dimensional film thickness we had

roughly 20 by 30 micron and 2 by 3 the 0.66 value at the exit and at the entrance it was a four by 3 it was 1.33.

Now, what we are saying here in that film thickness this will be added. So, whatever the two by three in that case the film thickness say this expression because this is going to add because of deformation, two surfaces are going to get most sufficient due to deformation and account this kind of film thickness variation.

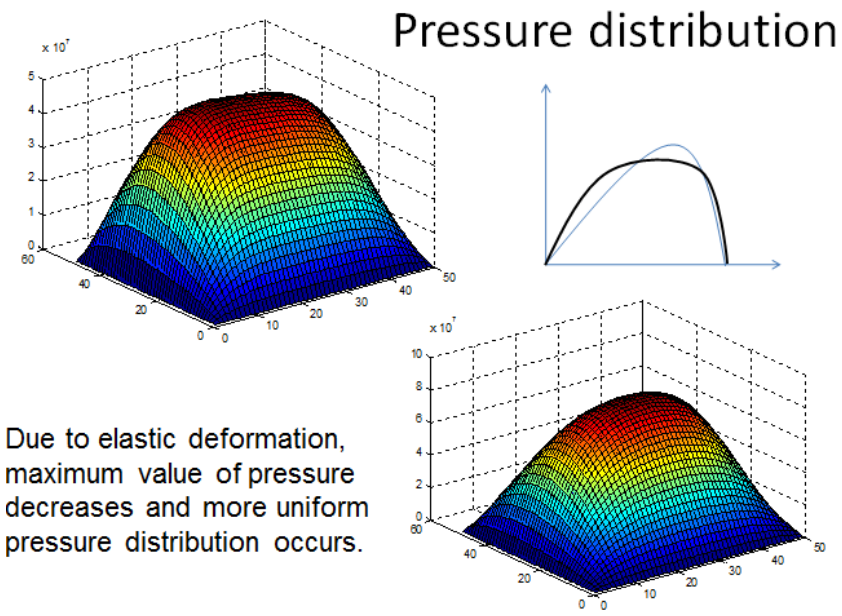
Now, what will happen if I increase the film thickness in any point? Pressure is going to go down, pressure will decrease and we require really iterative loop to come with a final results. What I am trying to say, use hydro dynamic lubrication mechanism, use that algorithm find out the pressure at every node, from those pressure find out the deformation of the every node, redo it. Now, you can come back, calculate the film thickness at every node, use those film thicknesses in the expression to find out hydro dynamic pressure, from hydro dynamic pressure again find out what will be deflection. And to do it need to go through a number of iterations to find out the final conversion solution.

Or what we are saying initially assuming elastic deformation zero; after that we know what will be the hydro dynamic lubrication film thickness and using this expression we find out pressure, after finding the pressure, we find this elastic deformation, we go back and change the film thickness expression.

And again redo it, again find out the hydro dynamic pressure, from hydro dynamic pressure again you find out the delta, from delta again come back to this equation, find out the film thickness a couple of iterations, not couple of iteration may be say hundred iteration plus it may give some solution and that is given over here. When we are not accounting iterations, we are not accounting a change in geometry and after evaluating the pressure and here we are going through number of iterations and this is a final curve.

What we are able to see here the deformation after a number of iteration is again reduced. That is clearly the pressure will come down, reduction in pressure will reduce a deflection and that is why the overall deformation will turn out to be lesser. You can see here it is crossing the four and reaching may be say 5.2 or something like that .052 while here it is a much lesser than that even 0.038. So, there is a variation. What does this mean? That we need to account this variation in film thickness, So that we can find out the right results otherwise it will be problematic.

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Now if we account the pressure distribution with elastic deformation; accounting elastic deformation what we are going to get is something like this. Pressure distribution something like this shows a three point something as the pressure. And if we do not account pressure elastic deformation use only the pressure viscosity relation. Do not account pressure deformation of the surfaces we are going to get much different results. You can see here when we are not accounting elastic deformation pressure shown over here is something more than four. When we are accounting pressure deflection due to pressure the pressure has come down.

Or another word; we need to account this kind of elastic deformation or we need to go through an elastohydro dynamic lubrication to find out the right results, the right pressure distribution because the pressure distribution is a first point to find out the friction losses also.

If we do not account pressure, right pressure distribution, the friction losses will be different, mass flow will be different and most of the parameter which are dependent on the pressure will turn out to be different. It will not give accurate results. So, as far as possible if the pressure is high and this is going to the find the elastic deformation or its going to deform elastically that surface we need to account.

What we say due to elastic deformation, the maximum value of pressure decreases and more uniform pressure distribution occurs.

See there is this stretch much larger than this stretch or there is more uniform distribution. That means, load carrying capacity is not going to decrease load carrying capacity of the surfaces will remain same only the maximum pressure is going to decrease. And more surfaces are going to sustain same pressure. Or in other words if I say this is without elastic deformation. If I account elastic deformation, pressure distribution will change something like this.

However, area under this curve is not going to change. That is what the load carrying capacity of this will remain same only. What is happening? The maximum peek pressure here is different than peek pressure over here or pressure is going to decrease in this case. So, this is a recommended case we say the elastic deformation of the surface is needed to be accounted properly. If we do not account we may not get good results.

So, we will continue with full film lubrication mechanism. We will be considering energy equation because we have considered yet only elementary treatment of thermal heating. While we are going to consider now in a slightly deeper or more description will be given in my next lecture.

So, most of the next lecture will be more focused on the energy equation, how we are going to account the temperature difference, how we are going to evaluate the temperature difference and how the temperature difference is going to change in the load carrying capacity. Thank you.